JHEP HYPER VERSION

Received: *January 29, 1999*, Accepted: *May 3, 1999*

Non–trivial flat connections on the 3–torus I: G_2 and the orthogonal groups

Arjan Keurentjes

Instituut-Lorentz for theoretical physics, Universiteit Leiden
P.O. Box 9506, NL-2300 RA Leiden, The Netherlands
E-mail: arjan@lorentz.leidenuniv.nl

Abstract: We propose a construction of non-trivial vacua for Yang-Mills theories on the 3–torus. Although we consider theories with periodic boundary conditions, twisted boundary conditions play an essential auxiliary role in our construction. In this article we will limit ourselves to the simplest case, based on twist in SU(2) subgroups. These reproduce the recently constructed new vacua for SO(N) and G_2 theories on the 3–torus. We show how to embed the results in the other exceptional groups F_4 and $E_{6,7,8}$ and how to compute the relevant unbroken subgroups. In a subsequent article we will generalise to SU($N > 2$) subgroups. The number of vacua found this way exactly matches the number predicted by the calculation of the Witten index in the infinite volume.

Keywords: Gauge Symmetry, Supersymmetry Breaking.

JHEP05(1999)001

Contents

1. Introduction

When constructing gauge field theories on a torus, a logical starting point is to begin with the construction of the classical vacua. For a gauge theory, this means finding all solutions with vanishing field strength, $F_{\mu\nu} = 0$, while respecting the appropriate boundary conditions. For Yang-Mills theories the relevant equation is non-linear, and hence non-trivial. With periodic boundary conditions, one trivial class of solutions is easily found, one can set $A_\mu = c^a_\mu H_a$ with c^a_μ a constant, and H_a group generators in a maximal abelian algebra of the group (Cartan subalgebra (CSA)). For SU(N) and Sp(N), all vacuum solutions are gauge equivalent to a member of this class, but in general this turns out not to be the case.

This problem is of relevance for the calculation of the Witten index, $\mathrm{Tr}(-1)^F$ in 4-dimensional supersymmetric gauge theories (where F stands for fermion number, and the trace is over the Hilbert space). In a classic paper [1], Witten proposed to formulate the theory on $T^3 \times \mathbb{R}$ with periodic boundary conditions (For SU(N),

$\mathrm{Tr}(-1)^F$ was also computed on $T^3 \times \mathbb{R}$ with twisted boundary conditions, but this approach cannot be applied to a general group). The expectation is, that for a sufficiently small size of the torus, a semiclassical calculation will be reliable. The energy spectrum is discrete, the vacua are distinct, and their contribution to $\mathrm{Tr}(-1)^F$ can in principle be counted. Based on the above mentioned trivial class of solutions, Witten predicted $\mathrm{Tr}(-1)^F = r + 1$ for theories with a simple gauge group, where r is the rank of the gauge group. It is argued that $\mathrm{Tr}(-1)^F$ is invariant under a large class of perturbations, and one can ask the question whether this result can be reproduced in the infinite volume limit, the theory on $\mathbb{R}^4$. In the infinite volume, the gauge theory exhibits chiral symmetry breaking via the axial anomaly, and, assuming gluino condensation, one finds h distinct vacua [1, 2] (h is the dual Coxeter number, which is equal to the Dynkin index of the adjoint representation, in an appropriate normalisation). The argument therefore suggests that $h = r + 1$ for simple gauge groups. This equality involves two purely group-theoretical quantities, and is easily checked. One finds that the equality is satisfied for the unitary and symplectic theories, but *not* for the theories based on orthogonal or exceptional gauge groups.

A resolution to this paradox was offered in [3]. Using an M-theory construction, Witten was able to show that in $\mathrm{SO}(N)$ with $N \geq 7$, flat connections exist which are not of the trivial type. These non-trivial flat connections give an extra contribution to $\mathrm{Tr}(-1)^F$ of $r' + 1$ where r' is the rank of the subgroup that is unbroken by the holonomies (Wilson loops along the non-trivial cycles of the torus). On the basis of his analysis Witten suggested that

$$h = \sum_i (r_i + 1)\,, \tag{1.1}$$

where the sum runs over disconnected components of moduli space, i.e. different classes of solutions, and r_i is the rank of the unbroken gauge group in each of these components. For $\mathrm{SU}(N)$ and $\mathrm{Sp}(N)$, the moduli space has only one component, and (1.1) reduces to the old formula. For the orthogonal groups, the extra vacua of Witten make up a second component in moduli space, and the result satisfies the equality (1.1). Witten conjectured that extra solutions might also solve the problem for the exceptional groups. Indeed, for G_2, it was shown in [4] that an extra vacuum exists, and (1.1) is satisfied. We will describe a method that allows us to address the question for the other exceptional groups.

The methods involved so far to find extra vacua rely heavily on the properties of orthogonal groups [3, 4]. The G_2 case is solvable because of the relation of G_2 to $\mathrm{SO}(7)$. In this paper, a more systematic way of constructing non-trivial vacua will be discussed. The existence of extra vacua in G_2 and $\mathrm{SO}(N)$ will be linked to the fact that these groups have special subgroups. This allows us to reconstruct the holonomies explicitly. Our construction can easily be embedded in the exceptional groups F_4 and $E_{6,7,8}$. In this paper we will restrict ourselves to the simplest

JHEP05(1999)001

realisation of our construction, based on twist in SU(2)-subgroups. This will be sufficient to reproduce the previous results from [3, 4] for G_2 and the orthogonal groups. In a subsequent article [5] we will show that constructions with twist in SU(N)-subgroups (with $N > 2$) can be realised in the other exceptional groups (and only in these). Although we do not know how to prove whether our method gives a complete classification of the flat connections for exceptional gauge groups on T^3, a strong indication that this is the case, is the fact that (1.1) is satisfied when the new vacua are included.

The outline of this paper will be as follows. In section 2 we will introduce our methods, and some machinery. In the next section we will apply our construction to find the new vacua, and calculate the unbroken gauge group. In an appendix we briefly review Lie algebras of simple Lie groups to outline our conventions. In a subsequent article [5] we will construct more vacua for the exceptional groups, and demonstrate how the Witten index problem is solved. These results are summarised in two tables.

2. Non-trivial flat connections

2.1 Holonomies and vacua

For any flat periodic connection on a torus, one can define a set of holonomies, Wilson loops along nontrivial cycles of the torus,

$$\Omega_k \;=\; P\exp\left\{i\int_0^{L_k} A_k(x)dx^k\right\}, \tag{2.1}$$

where $k = 1, 2, 3$ labels the holonomy corresponding to the cycle wrapping around the x, y, z direction respectively, L_k is the length of the cycle.

The fact that the connection is flat will imply that the holonomies commute. If one is only working with fields that take values in a representation of the gauge group that does not faithfully represent the center (implying that the representation is not simply connected), it is possible to impose commutativity up to an element of the center [6]. Working in a representation of the group that is simply connected, one can easily distinguish between commutativity, and commutativity up to a center element (twisted boundary conditions). We mention these points because twisted boundary conditions will form an essential ingredient in what follows, although in a somewhat unexpected way. As shown in [4], the fact that holonomies commute in a simply connected representation, is sufficient for the existence of a corresponding flat connection. Thus, we can use the holonomies (2.1) to characterise the flat connections.

Under periodic gauge transformations, the holonomies transform covariantly ($\Omega_k' = g\,\Omega_k g^{-1}$), and hence their traces are gauge invariant. For the trivial class of solutions $A_\mu = c^a_\mu H_a$, the holonomies $\Omega_k = \exp(ic^a_k H_a L_k)$ are on the so-called maximal torus (obtained by exponentiating the CSA). To find a different solution, one has to find holonomies that commute, but do not lie on a maximal torus. The corresponding flat connections are no longer of a simple form in these cases.

For T^2, using the complex structure on the 2-torus, it is possible to prove (as sketched in a footnote in [3]) that the moduli space of flat connections is connected, and hence the trivial class is the only class of solutions . For T^3 this is not the case, but using the result for T^2 one can arrange that any two out of the three holonomies are exponentials of elements of the CSA. The third holonomy is therefore the crucial one: either it can be written as the exponent of an element of the CSA, and we have a trivial solution, or it cannot, and one has a non-trivial solution.

The local parameters of the moduli space can be found by perturbing the holonomies Ω_k around a solution, demanding they still commute (so that the perturbations also lead to admissible vacua). An infinitesimal perturbation of the holonomy looks as follows (with T^a the group generators)

$$\Omega'_k = \Omega_k(1 + i\alpha^a_k T^a)\,. \tag{2.2}$$

When we require that all Ω'_k still commute, this implies the conditions [4]

$$\alpha^a_k \Omega_k [T^a, \Omega_l] = \alpha^a_l \Omega_l [T^a, \Omega_k]\,. \tag{2.3}$$

If there are generators T^a that commute with each of the Ω_k, this equation does not lead to any restrictions on the corresponding α^a_k, and these generators correspond to admissible perturbations. They will generate a group which is unbroken by the holonomies, and this is the group that is relevant in the calculation of $\mathrm{Tr}(-1)^F$. We now show that the generators that do not commute with all holonomies correspond to global gauge transformations, for Ω_k based on the $\mathrm{SU}(N)$ subgroup construction to be described (for $N > 2$ see ref. [5]). From this construction it follows that one can choose a basis T^a for the Lie algebra such that

$$\mathrm{Ad}(\Omega_k)T^a = \Omega_k T^a \Omega_k^{-1} = \exp\left(i\frac{2\pi n^a_k}{N}\right) T^a \qquad n^a_k, N \in \mathbb{Z} \qquad \forall k, a\,. \tag{2.4}$$

In that case the condition (2.3) reads

$$\alpha^a_k \left(\exp\left(-i\frac{2\pi n^a_l}{N}\right) - 1\right) \Omega_k \Omega_l T^a = \alpha^a_l \left(\exp\left(-i\frac{2\pi n^a_k}{N}\right) - 1\right) \Omega_k \Omega_l T^a\,, \tag{2.5}$$

Restricting ourselves to generators that do not commute with all holonomies, we define for k such that $[\Omega_k, T^a] \neq 0$

$$\beta^a = \frac{\alpha^a_k}{1 - \exp(-i\frac{2\pi n^a_k}{N})}\,. \tag{2.6}$$

Equation (2.5) implies that β^a is independent of k, whereas (2.3) implies $\alpha_k^a = 0$ if $[\Omega_k, T^a] = 0$, so we can write

$$i\alpha_k^a \Omega_k T^a = i\beta^a [\Omega_k, T^a]\,. \tag{2.7}$$

We are thus left with a global gauge transformation, generalising a result derived in [4] for SO(7).

2.2 Method of construction

Our construction is based on a variant of the so-called multi-twisted boundary conditions, as considered first in [7] involving subgroups of the gauge group.

We restrict ourselves to Yang-Mills theories with a compact, simple and simply connected gauge-group G. We will be interested in subgroups PG'_N whose universal covering $\widetilde{PG_N}'$ is the product of N factors $\tilde{G}'$. The representation of PG_N will in general not be irreducible, nor are its irreducible components in the same congruence class.

The *global* structure of our subgroup PG'_N will *not* be that of a direct product group. For the realisation of twisted boundary conditions, it is necessary that a non-trivial discrete central subgroup has been divided out; for multi-twisted boundary conditions this discrete central subgroup is diagonal. As a relevant example of such a subgroup, consider SO(4) which is locally $\mathrm{SU}(2) \times \mathrm{SU}(2)$, but has global structure $(\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the diagonal subgroup in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ centre of $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

For the discussion here we will restrict ourselves to a subgroup PG'_2 (with universal covering $\tilde{G}'^2$), since the generalisation to PG'_N will be obvious. We assume that $\tilde{G}'$ allows a non-trivial center Z, and that the global subgroup PG'_2 is a representation of $(\tilde{G}' \times \tilde{G}')/Z_{diag}$ where $Z_{diag} \cong Z$ is the diagonal subgroup of the centre $Z \times Z$ of $\tilde{G}'^2$. Now select two elements P_1, Q_1 generated by the Lie algebra of the first $\tilde{G}'$ factor ($\tilde{G}'_1$), and two elements P_2, Q_2 generated by the Lie algebra of the second $\tilde{G}'$ factor ($\tilde{G}'_2$), such that they commute up to a non-trivial element of the center Z of $\tilde{G}'_i$:

$$P_1 Q_1 = z_1 Q_1 P_1 \tag{2.8}$$

$$P_2 Q_2 = z_2 Q_2 P_2\,. \tag{2.9}$$

For irreducible representations of $\tilde{G}'_i$, z_i is a root of unity, for a reducible representation z_i is a diagonal matrix, with on the diagonal the center elements appropriate for the different irreducible components. Since we only deal with representations of $(\tilde{G}' \times \tilde{G}')/Z_{diag}$, z_1 and z_2 are actually elements of the central subgroup $(Z \times Z)/Z_{diag} \cong Z$. By picking P_i and Q_i in a specific way, we can thus arrange that (2.8), (2.9) are satisfied with the additional condition

$$z_1 = (z_2)^{-1} \equiv z\,. \tag{2.10}$$

We now have

$$P = P_1 P_2\,, \qquad Q = Q_1 Q_2\,, \quad \Rightarrow \quad PQ = QP\,. \tag{2.11}$$

It is possible[1] and convenient to embed the PG_2' subgroup in such a way that its maximal torus T' is a subgroup of a maximal torus T of G. If the PG_2'-subgroup has the same rank as G, then the tori T and T' coincide. If this is not the case, then there are multiple ways to extend T' to a maximal torus T of G.

We can choose the elements P_1 and P_2 to lie on the torus T'. Since the maximal torus is abelian, it immediately follows that neither Q_1 nor Q_2 is on T', and neither is their product Q (since Q does not commute with either of the P_i). We will now construct a third element P' by "twisting" one of the G''s with respect to the other G'

$$P' = Q_1^n P Q_1^{-n} = Q_1^n P_1 Q_1^{-n} P_2 = P_1 Q_2^{-n} P_2 Q_2^n = z^{-n} P\,. \tag{2.12}$$

We can define a set of holonomies by setting $\Omega_1 = P$, $\Omega_2 = P'$ and $\Omega_3 = Q$. We will always assume P' and P to be different, which is essential for finding non-trivial vacua. This limits n to a finite set, since there exists some n for which $z^{-n} = 1$. We also should not allow z to be an element of the centre of G, since this will also imply that the connection defined by the holonomies P, P' and Q is trivial (we can arrange that P and Q are on a maximal torus of G, and then, since the centre of G is also on this maximal torus, P' must be on this maximal torus). Since P is on the maximal torus T', and z is an element of the centre of PG_2', and the center is generated by the CSA, P' is on the maximal torus T'. By assumption the torus T' is contained in a maximal torus T of G. To define a non-trivial flat connection Q should not be on any maximal torus T of G. If Q is on a maximal torus of G, then the maximal abelian subgroup that commutes with P, P' and Q is a maximal torus.

Since the construction involves subgroups $\tilde{G}'$ with a non-trivial center, one may always take a subgroup of $\tilde{G}'$ that is a product of unitary groups[2]. Henceforth, we shall allways assume $\tilde{G}'$ to be a product of $\mathrm{SU}(N)$'s. Thus we may take $\tilde{G}' = \mathrm{SU}(N_1)^{n_1} \times \mathrm{SU}(N_2)^{n_2} \times \cdots$ (N_i different, n_i positive integers), which has as center $(\mathbb{Z}_{N_1})^{n_1} \times (\mathbb{Z}_{N_2})^{n_2} \times \cdots$.

2.3 Diagonal subgroups

Although not strictly necessary, extremely useful for our calculations is the concept of a diagonal subgroup. It is possible to construct a diagonal subgroup D (with universal covering $\tilde{D} = \tilde{G}'$) in PG_N' as follows: construct a Lie algebra $\mathcal{L}$ for $\tilde{G}'$, consisting of elements T^a. Then the Lie algebra for (PG_N' has the structure $\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_N$ with each of the $\mathcal{L}_i \cong \mathcal{L}$. Hence we can write T_i^a, for the generator from $\mathcal{L}_i$ that

[1] It was proven by Dynkin [8] that if a Lie algebra $\mathcal{L}_G$ has a subalgebra $\mathcal{L}_{G'}$, then the CSA of $\mathcal{L}_G'$ can be chosen to be contained in the CSA of $\mathcal{L}_G$ (by applying a suitable automorphism of $\mathcal{L}_G$). Upon exponentiation, one finds that the maximal torus of the group G', generated by $\mathcal{L}_{G'}$, is contained in the maximal torus of the group G, generated by $\mathcal{L}_G$.

[2] If some simple subgroup with non-trivial center is not $\mathrm{SU}(N)$, its center is either $\mathbb{Z}_2$ ($\mathrm{Sp}(n)$, $\mathrm{SO}(2n+1)$ and E_7), $\mathbb{Z}_4$ ($\mathrm{SO}(4n+2)$) , $\mathbb{Z}_2 \times \mathbb{Z}_2$ ($\mathrm{SO}(4n)$) of $\mathbb{Z}_3$ (E_6). In these cases the center is contained in an $\mathrm{SU}(2)$, $\mathrm{SU}(4)$, $\mathrm{SU}(2) \times \mathrm{SU}(2)$ or $\mathrm{SU}(3)$-subgroup

corresponds to T^a under an isomorphism mapping $\mathcal{L}$ to $\mathcal{L}_i$. The diagonal subgroup of PG'_N is then constructed by taking as generators $T_1^a + T_2^a + \cdots + T_N^a$. This construction is not unique, there are many isomorphisms from $\mathcal{L}_i$ to $\mathcal{L}$, and these will give different (but isomorphic) diagonal subgroups.

We now take P_1, P_2, Q_1, Q_2 as before. The P_i are elements of the maximal torus of $\tilde{G}'_i$, so we can write $P_i = \exp(ih_i)$ with h_i an element of the CSA of $\mathcal{L}_i$, and similarly $Q_i = \exp(ie_i)$ with e_i *not* in the CSA. The elements $P = P_1P_2 = \exp(i(h_1+h_2))$ and $Q = Q_1Q_2 = \exp(i(e_1+e_2))$ are then elements of a diagonal group D, as constructed in the above. Conjugating with Q_1^n will produce $Q_1^n P Q_1^{-n} = P'$ and $Q_1^n Q Q_1^{-n} = Q$, which are elements of a diagonal subgroup D', isomorphic to D (the isomorphism being given by $D \leftrightarrow Q_1^n D Q_1^{-n}$).

3. Non-trivial vacua based on twist in SU(2)

In this section we specialise to $\tilde{G}' = \mathrm{SU}(2)$. We will start by developing the relevant tools for this subgroup. After that we will give an overview of groups in which our construction can be realised. These include $\mathrm{SO}(N)$ where the result from [3] is reproduced, and G_2, where we rederive the result from [4].

3.1 Twist in SU(2)

We will use the following convention for the su(2) algebra:

$$[L_3, L_+] = L_+\,, \qquad [L_3, L_-] = -L_-\,, \qquad [L_+, L_-] = 2L_3\,. \tag{3.1}$$

We have $L_3^\dagger = L_3$ and $L_+^\dagger = L_-$. With these conventions the eigenvalues of L_3, $(L_+ + L_-)/2$ and $i(L_+ - L_-)/2$ are half-integers for representations that exponentiate to SU(2), and integers for representations that exponentiate to SO(3).

In SU(2) we will be looking for elements p and q such that

$$pq = -qp\,. \tag{3.2}$$

The standard convention is to take:

$$p = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \exp\frac{i\pi}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad q = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \exp\frac{i\pi}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.3}$$

In terms of generators this is $p = \exp(i\pi L_3)$ and $q = \exp(i\pi(L_+ + L_-)/2)$ (When lifted to SO(3), the elements are $p = \mathrm{diag}(-1,-1,1)$ and $q = \mathrm{diag}(1,-1,-1)$, which commute). The commutation relations of p, q with the algebra elements are easily determined

$$\begin{aligned} pL_3 &= L_3p & qL_3 &= -L_3q \\ pL_+ &= -L_+p & qL_+ &= L_-q \\ pL_- &= -L_-p & qL_- &= L_+q\,. \end{aligned} \tag{3.4}$$

Note that q induces the Weyl reflection on the root lattice. This has a nice analogon for twist in $\mathrm{SU}(N), N > 2$ [5].

The elements P_i will take the role of p in the above, the elements Q_i take the role of q. Now notice that the condition (2.10) implies that the diagonal group D that contains $P = P_1 P_2$ and $Q = Q_1 Q_2$ is actually an SO(3) (it has a trivial center). This can be seen as follows: the diagonal subgroup-construction provides a homomorphism from any of the $G' \cong \mathrm{SU}(2)$ factors to the diagonal subgroup D. Under this homomorphism the non-trivial center element of SU(2) is mapped to the identity in D (because $P_i \to P$, $Q_i \to Q$, we have $P_i Q_i P_i^{-1} Q_i^{-1} = z \to PQP^{-1}Q^{-1} = 1$).

3.2 Calculation of the unbroken subgroup

One of the issues we did not address so far is the fact that we require the holonomies Ω_i (P, P' and Q) to commute in a simply connected representation (otherwise the theorem of [4] might not work). In fact we will show that they commute in *any* representation, which seems more general, but is equivalent to the previous statement by the theory of compact Lie groups. For the SU(2)-based construction described in this paper, a sufficient condition for the commuting of all holonomies is that D is an SO(3)-group (since this will imply that P and Q commute, and from this it follows that P' and Q commute. P and P' commute by construction). To determine whether D is an SO(3) subgroup, we construct its algebra.

As remarked in footnote 1, it is always possible to choose an embedding of a subgroup such that its CSA is contained in the CSA of the group that contains the subgroup, so let $L_3 = h_\zeta$ for some h_ζ in the CSA of G. The eigenvalues of h_ζ are determined by taking inner products with the weights of G. We want L_3 to be an SO(3)-generator, and therefore its eigenvalues should be integers. Therefore $\langle \lambda, \zeta \rangle$ should be integer for any weight λ. Since any weight is expressible as a linear combination of fundamental weights and simple roots with integer coefficients, the condition that L_3 should have integer eigenvalues can be translated to

$$\langle \alpha_i, \zeta \rangle \in \mathbb{Z} \tag{3.5}$$

$$\langle \Lambda_i, \zeta \rangle \in \mathbb{Z}\,, \tag{3.6}$$

where α_i are the simple roots of G, and Λ_i are its fundamental weights. However, if the first of these two conditions is satisfied, then the second condition is satisfied for some weights, namely those fundamental weights with $\Lambda_i = \sum_k q_k \alpha_k$ with q_k integer (we call these integer weights). Hence the second condition only has to be checked for what we will call non-integer weights. A list of these is contained in our appendix.

We will want to compute products like pTp^{-1}, qTq^{-1}, where T are generators of the group G we are working in, and p and q are as in the previous paragraph. We will be looking at SO(3)-subgroups, and the generators of G split into SO(3) representations. pTp^{-1} is easily calculated. With $L_3 = h_\zeta$ we have

$$ph_\alpha p^{-1} = h_\alpha\,, \qquad pe_\alpha p^{-1} = \exp(i\pi\langle\alpha,\zeta\rangle)e_\alpha\,. \tag{3.7}$$

Note that this implies that the generators either commmute or anticommute with p.

There is also a nice and easy way to compute qTq^{-1}. First decompose the representation of G into irreducible representations of SO(3). In each irrep, construct the normalised eigenvectors ψ_λ of L_3: $L_3\psi_\lambda = \lambda\psi_\lambda$. It then follows that

$$L_3 q\psi_\lambda = -qL_3\psi_\lambda = -\lambda q\psi_\lambda\,. \tag{3.8}$$

Hence we conclude $q\psi_\lambda = \phi_\lambda\psi_{-\lambda}$ (where ϕ_λ is a phase factor). In fact, since we are only dealing with SO(3) representations of q, we know that the eigenvalues of q should be ± 1 and hence $q^2 = 1$, meaning that $\phi_\lambda\phi_{-\lambda} = 1$. Moreover from $qL_+ = L_- q$ we easily find that[3]

$$\phi_\lambda\psi_{-(\lambda+1)} = \phi_{\lambda+1}\psi_{-(\lambda+1)}\,. \tag{3.9}$$

So, $\phi_\lambda = \phi_{\lambda+1} = \phi$, independent of λ, and $\phi^2 = 1$. It is now trivial to construct eigenvectors and eigenvalues for q:

$$\begin{aligned} \psi_\lambda + \psi_{-\lambda} &\rightarrow \text{eigenvalue } \phi \\ \psi_\lambda - \psi_{-\lambda} &\rightarrow \text{eigenvalue } -\phi\,. \end{aligned} \tag{3.10}$$

Now remember once more that only SO(3) representations occur, meaning that $\lambda \in \{0, \pm 1, \pm 2, \ldots\}$. If the dimension of our representation is $2n+1$, then the eigenvalue ϕ occurs $n+1$ times, and $-\phi$ n times. The determinant of the matrix is then $(-)^n\phi^{2n+1} = (-)^n\phi$. The determinant should however be 1 since the matrix has been obtained by exponentiating a traceless generator. Hence we find $\phi = (-)^n$, and the action of q on any representation of SO(3) is fully determined, and since by assumption G splits into SO(3) irreps, the action of q in G is completely determined. Most of the above is also valid for SU(2)-irreps, but there are two differences: one finds $\phi^2 = -1$, and it is not possible to determine whether $\phi = i$ or $-i$. This ambiguity comes from the center of SU(2), which we are unable to detect since we are trying to determine q from commutation relations alone. Notice that these considerations imply that for an appropriately chosen basis of generators T of G that (compare to (2.4))

$$pT^a p^{-1} = \pm T^a\,, \qquad qT^a q^{-1} = \pm T^a\,. \tag{3.11}$$

[3] We use the Condon-Shortley phase convention: $L_+\psi^j_\lambda = \sqrt{(j-\lambda)(j+\lambda+1)}\psi^j_{\lambda+1}$ and $L_-\psi^j_\lambda = \sqrt{(j+\lambda)(j-\lambda+1)}\psi^j_{\lambda-1}$, where j is the usual angular momentum number, related to the dimension d of the representation by $j = (d-1)/2$

JHEP05(1999)001

3.3 Realisations of the SU(2)-based construction

We will now discuss the cases in which our construction actually gives a non-trivial flat connection. Our conventions concerning roots and weights can be found in our appendices. For the decomposition of groups into subgroups, use was made of [9].

Although in each case our construction can be carried out in a subgroup $PG'_2 = \mathrm{SO}(4)$, we will often take PG'_N with $\widetilde{PG'_N} = \mathrm{SU}(2)^N$ where $N > 2$. This allows us to choose PG'_N to be a regular subgroup [8], that is, the root-lattice of PG'_N is a sublattice of the root-lattice of G. This gives an enormous simplification of the calculations. Our methods are not limited to regular subgroups, and we have actually carried out our construction for several irregular embeddings, but we always found the same results as for the regular embeddings. Therefore we will describe only constructions with regular embeddings.

Case of G_2. Our first example will be the non-trivial flat connection in G_2, described in [4]. We will treat this example in full detail to clarify our methods. G_2, being the group of lowest rank that posseses non-trivial flat connections on T^3, is the simplest from the point of view of our construction (unlike the constructions in [3, 4] that are simpler for orthogonal groups).

G_2 posseses an $\mathrm{SO}(4) \cong (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2$ subgroup. The first SU(2)-factor can be taken to be generated by $h_{\alpha_1}, e_{\alpha_1}$ and $e_{-\alpha_1}$, the second one is then generated by $h_{\alpha_1+2\alpha_2}, e_{\alpha_1+2\alpha_2}$ and $e_{-(\alpha_1+2\alpha_2)}$. We label the two SU(2) factors $\mathrm{SU}(2)_{1,2}$, and normalise their generators such that they satisfy the algebra (3.1):

$$\begin{array}{ll} \mathrm{SU}(2)_1 \;\times & \mathrm{SU}(2)_2 \\ l^1_3 = 4h_{\alpha_1} & l^2_3 = 12h_{\alpha_1+2\alpha_2} \\ l^1_+ = 2e_{\alpha_1} & l^2_+ = 2\sqrt{3}e_{\alpha_1+2\alpha_2} \end{array} \tag{3.12}$$

The diagonal subgroup is now easily constructed, being generated by

$$\begin{aligned} L^D_3 &= l^1_3 + l^2_3 = h_{16\alpha_1+24\alpha_2} \\ L^D_+ &= l^1_+ + l^2_+ = 2e_{\alpha_1} + 2\sqrt{3}e_{\alpha_1+2\alpha_2} \end{aligned} \tag{3.13}$$

The diagonal SU(2) turns out to be an SO(3). To check this we construct the weights for the su(2)-representation. These are $\langle 16\alpha_1 + 24\alpha_2, \lambda\rangle$ where λ are the weights of G_2. However, since any weight of G_2 is of the form $p\alpha_1 + q\alpha_2$, with p, q integer, it is sufficient to compute $\langle 16\alpha_1 + 24\alpha_2, \alpha_1\rangle = 1$ and $\langle 16\alpha_1 + 24\alpha_2, \alpha_2\rangle = 0$. We find that the weights of the su(2) are always integer, no matter what representation of G_2 we use, and hence the su(2) is actually an so(3). We find the following decompositions for the fundamental and adjoint irreps:

$$\begin{array}{rcccl} G_2 & \to & \mathrm{SU}(2)\times\mathrm{SU}(2) & \to & \mathrm{SU}(2)/\mathbb{Z}_2 \cong \mathrm{SO}(3) \\ \mathbf{7} & \to & (\mathbf{1},\mathbf{3}) \oplus (\mathbf{2},\mathbf{2}) & \to & \mathbf{3}\oplus\mathbf{3}\oplus\mathbf{1} \\ \mathbf{14} & \to & (\mathbf{3},\mathbf{1}) \oplus (\mathbf{2},\mathbf{4}) \oplus (\mathbf{1},\mathbf{3}) & \to & \mathbf{3}\oplus\mathbf{5}\oplus\mathbf{3}\oplus\mathbf{3}\,. \end{array} \tag{3.14}$$

JHEP05(1999)001

It is easy to construct the elements P and Q: Take

$$P = \exp\left(i\pi L_3^D\right) \qquad Q = \exp\left(i\frac{\pi}{2}\left(L_+^D + L_-^D\right)\right). \tag{3.15}$$

Now we wish to obtain P'. Conjugation with $Q_1 = \exp(i\frac{\pi}{2}(l_+^1 + l_-^1))$ generates the Weyl reflection in the first of the two SU(2)-factors:

$$l_3^1 \leftrightarrow -l_3^1\,, \qquad l_+^1 \leftrightarrow l_-^1\,. \tag{3.16}$$

This will take the diagonal subgroup D to a diagonal subgroup D' generated by:

$$\begin{aligned} L_3^{D'} &= -l_3^1 + l_3^2 = h_{8\alpha_1+24\alpha_2} \\ L_+^{D'} &= \;\; l_-^1 + l_+^2 \;= 2e_{-\alpha_1} + 2\sqrt{3}e_{\alpha_1+2\alpha_2}\,. \end{aligned} \tag{3.17}$$

Since $L_+^{D'} + L_-^{D'} = L_+^D + L_-^D$, Q is also an element of D'. Constructing P' proceeds as in the above:

$$P' = \exp(i\pi L_3^{D'})\,. \tag{3.18}$$

P, P' and Q commute by construction. It is also clear that the flat connection implied by $\Omega_1 = P$, $\Omega_2 = P'$ and $\Omega_3 = Q$ is non-trivial, since the maximal torus of G_2 is simply the direct product of the tori of D and D', and Q is not on either one.

To calculate the unbroken subgroup, we calculate the commutators of generators with the Ω_i's. As explained before only generators that commute with all three Ω_i's will be relevant. Therefore the most efficient way to proceed is to first compute the commutators with $\Omega_1 = P$ and $\Omega_2 = P'$, since these are the easiest, and only compute the commutator with $\Omega_3 = Q$ for those generators that commute with both P and P'.

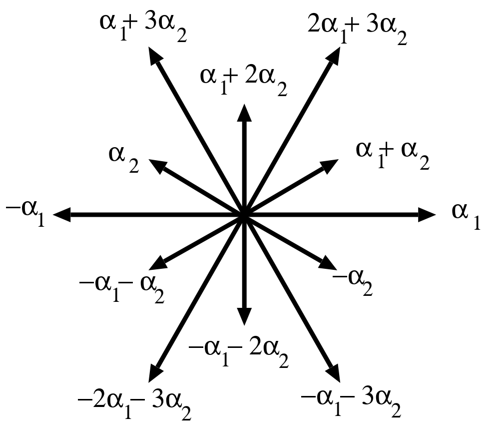

Figure 1: The root diagram for G_2.

We can use the results of section 3.2. If in this section we substitute L_3^D for L_3, and $L_\pm^D$ for $L_\pm$, then it is clear we should identify P with p, and Q with q. If in section 3.2 we substitute $L_3^{D'}$ for L_3, and $L_\pm^{D'}$ for $L_\pm$, then we should identify P' with p, and Q with q. For the commutators of the algebra with P, we use (3.7), with $\zeta = 16\alpha_1 + 24\alpha_2$. We find that the CSA commutes with P, and e_β commutes with P only for $\beta = \pm\alpha_2$ and $\beta = \pm(2\alpha_1+3\alpha_2)$. For the commutators of the algebra with P', we use (3.7) with $\zeta = 8\alpha_1 + 24\alpha_2$, from which it follows that e_β with $\beta = \pm\alpha_2$ and $\beta = \pm(2\alpha_1 + 3\alpha_2)$ anticommute with P'. Hence the only generators that commute with both P and P' are the CSA-generators h_β.

To determine the effect of conjugation with Q on these, we study the branching of $G_2 \to \mathrm{SO}(3)$, where the SO(3) is either D or D'. We will take D, so the ladder operators are $L^D_\pm$ and the CSA-generator is L^D_3. We work with the generators of G_2 themselves, and hence we are in the adjoint representation of G_2. Eigenvectors of L^D_3 are easily found, since $\mathrm{ad}(L^D_3)e_\beta = [h_{16\alpha_1+24\alpha_2}, e_\beta] = \langle 16\alpha_1 + 24\alpha_2, \beta\rangle e_\beta$ in our conventions, and , $\mathrm{ad}(L^D_3)h_\beta = 0$, h_β and e_β are eigenvectors of L^D_3. We can now split the representation in irreducible components, by the standard procedure of looking for highest eigenvalues, and then applying the ladder operators $L^D_\pm$ to complete a representation, constructing the orthogonal complement etc.. We find that both CSA-generators have eigenvalue 0 in a **3** irrep of SO(3). We should thus identify each CSA-generator to ψ_0 in a 3-dimensional representation of SO(3), and using the results of section 3.2 we find that, since $Q\psi_0 = -\psi_0$, $Qh_\beta Q^{-1} = \mathrm{Ad}(Q)h_\beta = -h_\beta$. The CSA generators thus anticommute with Q. Thus there is no generator that commutes with all Ω_i, and, as already established in [4], the vacuum implied by these holonomies is isolated, and there is only a discrete unbroken subgroup.

Finally we will give a matrix representation of the holonomies. The **7** irrep of G_2 has as its weights $0, \alpha_1+\alpha_2, \alpha_1+2\alpha_2, \alpha_2, -(\alpha_1+\alpha_2), -(\alpha_1+2\alpha_2), -\alpha_2$. Using this ordering of weights, we find:

$$\begin{aligned}
\Omega_1 &= P = \mathrm{diag}(1,-1,-1,1,-1,-1,1) \\
\Omega_2 &= P' = \mathrm{diag}(1,1,-1,-1,1,-1,-1) \\
\Omega_3 &= Q = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{3.19}$$

These can be brought to the form used in [4].

Case of SO(7). In SO(7), there exists a subgroup PG'_3 with $\widetilde{PG'_3} = \mathrm{SU}(2)^3$. The SU(2)-factors can be taken to be generated by subalgebras with elements $h_{\beta_i}, e_{\pm\beta_i}$, with $\beta_1 = (1,-1,0)/\sqrt{10}$ for the first factor, $\beta_2 = (1,1,0)/\sqrt{10}$ for the second factor, and $\beta_3 = (0,0,1)/\sqrt{10}$ for the third. The diagonal subgroup is then (appropriate normalisations for each generator included)

$$\begin{aligned}
L^D_3 &= l^1_3 + l^2_3 + l^3_3 = \sqrt{10}h_{(1,0,1)} \\
L^D_+ &= l^1_+ + l^2_+ + l^3_+ = \sqrt{5}e_{(1,-1,0)/\sqrt{10}} + \sqrt{5}e_{(1,1,0)/\sqrt{10}} + \sqrt{10}e_{(0,0,1)/\sqrt{10}}
\end{aligned} \tag{3.20}$$

Again this is always an so(3)-algebra, which can be easily verified by calculating the inner products of $\sqrt{10}(1,0,-1)$ with the simple roots and non-integer weights.

Vector, spin, and adjoint representation branch as follows:

$$\begin{array}{rcccl}
\mathrm{SO}(7) & \to & (\mathrm{SU}(2))^3 & \to & \mathrm{SO}(3) \\
\mathbf{7} & \to & (\mathbf{1},\mathbf{1},\mathbf{3})\oplus(\mathbf{2},\mathbf{2},\mathbf{1}) & \to & 2(\mathbf{3})\oplus\mathbf{1} \\
\mathbf{8} & \to & (\mathbf{1},\mathbf{2},\mathbf{2})\oplus(\mathbf{2},\mathbf{1},\mathbf{2}) & \to & 2(\mathbf{3})\oplus 2(\mathbf{1}) \\
\mathbf{21} & \to & (\mathbf{3},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{3},\mathbf{1}) & & \\
 & & \oplus(\mathbf{1},\mathbf{1},\mathbf{3})\oplus(\mathbf{2},\mathbf{2},\mathbf{3}) & \to & \mathbf{5}\oplus 5(\mathbf{3})\oplus\mathbf{1}
\end{array} \tag{3.21}$$

To construct D', twist the first factor with respect to the other two

$$l_3^1 \leftrightarrow -l_3^1\,, \qquad l_+^1 \leftrightarrow l_-^1\,,$$

which leads to

$$\begin{aligned}
L_3^{D'} &= l_3^1 + l_3^2 + l_3^3 = \sqrt{10}h_{(0,1,1)} \\
L_+^{D'} &= l_+^1 + l_+^2 + l_+^3 = \sqrt{5}e_{-(1,-1,0)/\sqrt{10}} + \sqrt{5}e_{(1,1,0)/\sqrt{10}} + \sqrt{10}e_{(0,0,1)/\sqrt{10}}
\end{aligned} \tag{3.22}$$

The set of holonomies is

$$\Omega_1 = P = \exp\left(i\pi L_3^D\right), \quad \Omega_2 = P' = \exp\left(i\pi L_3^{D'}\right), \quad \Omega_3 = Q = \exp\left(i\frac{\pi}{2}\left(L_+^D + L_-^D\right)\right). \tag{3.23}$$

In the vector representation, these are equivalent to the holonomies of Witten [3]. There is no generator of the algebra that commutes with all three Ω_i.

Two remarks are in place here. First, one might think that it is arbitrary which of the SU(2)-factors one chooses to twist (in the sense of eq. (2.12)). Indeed, twisting the first factor leaving the other two the same, as in the above, will give a result equivalent to twisting the second factor while leaving the other two. However, twisting the third factor with respect to the other two will not work: If one tries

$$l_3^3 \leftrightarrow -l_3^3 \qquad l_+^3 \leftrightarrow l_-^3\,,$$

one finds

$$P' = \exp(i\pi\sqrt{10}h_{(1,0,-1)}) = \exp(i\pi\sqrt{10}h_{(0,0,-2)})\; P\,.$$

But, noticing that $\sqrt{10}h_{(0,0,1)}$ is a (correctly normalised) generator of an integral su(2)-subalgebra of so(7), we easily find $\exp(i\pi\sqrt{10}h_{(0,0,-2)}) = \pm 1$ ($+1$ if the so(7)-algebra is in the same congruence class as the vector or adjoint representation, -1 if the so(7)-irrep is isomorphic to the spin representation). Hence P and P' differ only by an element of the centre of $Spin(7)$, and as explained in section 2.2, the connection implied by setting $\Omega_1 = P$, $\Omega_2 = P'$ and $\Omega_3 = Q$ is trivial.

As a second remark, we consider the embedding of G_2 in SO(7). The so(7)-root diagram fits into a cube. To see its g_2-subalgebra, project onto the plane orthogonal to a diagonal. Take as diagonal the direction $(-1, -1, 1)$. The roots $\beta_2 = (1,1,0)/\sqrt{10}$ and $\beta_3 = (0,0,1)/\sqrt{10}$ will coincide under this projection (they will both project to $(1, 1, 2)/3\sqrt{10}$), and we find that the SO(7) result can be understood from the G_2-result. This will be less relevant for the orthogonal groups, but important for an understanding of the exceptional groups.

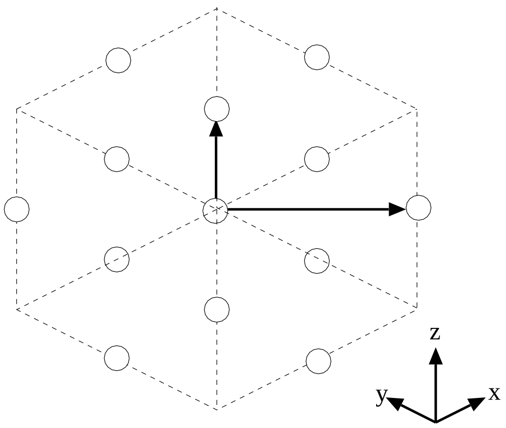

Figure 2: Projection of the roots of so(7) onto those of g_2

Case of SO(8)**.** In SO(8), there is a subgroup PG'_4, with $\widetilde{PG'_4} = \mathrm{SU}(2)^4$. All SU(2)-factors can be taken to be generated by algebras with elements $h_{\beta_i}, e_{\pm\beta_i}$, with $\beta_1 = (1,-1,0,0)/\sqrt{12}$, $\beta_2 = (1,1,0,0)/\sqrt{12}$, $\beta_3 = (0,0,1,-1)/\sqrt{12}$ and $\beta_4 = (0,0,1,1)/\sqrt{12}$. The diagonal subgroup is then

$$\begin{aligned} L_3^D &= \sqrt{12} h_{(1,0,1,0)} \\ L_+^D &= \sqrt{6}\left(e_{(1,-1,0,0)/\sqrt{12}} + e_{(1,1,0,0)/\sqrt{12}} + e_{(0,0,1,-1)/\sqrt{12}} + e_{(0,0,1,1)/\sqrt{12}}\right). \end{aligned} \tag{3.24}$$

It is easy to verify that this is an so(3)-algebra. Vector, and adjoint representation branch as follows:

$$\begin{array}{rlcl} \mathrm{SO}(8) & \rightarrow \qquad (\mathrm{SU}(2))^4 & \rightarrow & \mathrm{SO}(3) \\ \mathbf{8_v} & \rightarrow \quad (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}) & \rightarrow & 2(\mathbf{3}) \oplus 2(\mathbf{1}) \\ \mathbf{28} & \rightarrow (\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{1}) \oplus & & \\ & \quad \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3}) \oplus (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2}) & \rightarrow & (\mathbf{5}) \oplus 7(\mathbf{3}) \oplus 2(\mathbf{1}). \end{array} \tag{3.25}$$

Because of triality, the spin-representations $\mathbf{8_c}$ and $\mathbf{8_s}$ have a branching that differs only from the one for $\mathbf{8_v}$ in $\mathrm{SU}(2)^4$ by permutations of the $\mathbf{2}$'s and $\mathbf{1}$'s, their SO(3)-content is the same. Twist the first factor to construct D':

$$\begin{aligned} L_3^{D'} &= \sqrt{12} h_{(0,1,1,0)} \\ L_+^{D'} &= \sqrt{6}\left(e_{-(1,-1,0,0)/\sqrt{12}} + e_{(1,1,0,0)/\sqrt{12}} + e_{(0,0,1,-1)/\sqrt{12}} + e_{(0,0,1,1)/\sqrt{12}}\right). \end{aligned} \tag{3.26}$$

The holonomies are then constructed in the usual way (3.23). The unbroken subgroup is again discrete. The projection of SO(8) onto SO(7) is trivial, and we find that we can understand the SO(8)-result from the G_2-result.

Case of SO(N), $N > 8$. It is clear that the SO(7)-example and the SO(8)-example can both be embedded in SO(9). Notice however, that both in the embedding of the SO(7)-example, and the embedding of the SO(8)-example we find

$$L_3^D = \sqrt{12} h_{(1,0,1,0)}\,, \qquad L_3^{D'} = \sqrt{12} h_{(0,1,1,0)}\,.$$

The vectors $\sqrt{12}(1,0,1,0)$ and $\sqrt{12}(0,1,1,0)$ are called "defining vectors" for D and D', and there is a theorem by Dynkin [8], that two representations of SO(3) are equivalent, if their defining vectors are equivalent. Hence the SO(7) and SO(8) embeddings are equivalent, and will not lead to different results. One finds a U(1) $\cong$ SO(2) unbroken subgroup, but, as explained in [3, 4], the actual unbroken subgroup is O(2), because of discrete symmetries that are invisible in our approach.

In a similar way, the non-trivial flat connections for SO(N) with $N > 9$ can be constructed. For N sufficiently large, there are multiple ways of embedding an SU(2)^M with a diagonal SO(3). Although we know no simple way of proving this in our approach (their defining vectors need not be equivalent, for example), the analysis of [3] shows that these can never lead to new results, other than an SO(7) or SO(8) embedding will do. Note the chain of subgroups

$$\mathrm{SO}(N) \to \mathrm{SO}(N-1) \to \cdots \to \mathrm{SO}(7) \to G_2\,, \tag{3.27}$$

that shows that non-trivial flat connections in SO(N) can be derived from the SO(N)-subgroup G_2. The connected component of the maximal unbroken subgroup is SO($N-7$), as can be understood from the fact that SO(N) branches into $G_2 \times \mathrm{SO}(N-7)$.

Case of F_4. The easiest way to proceed in F_4 is by using the fact that the so(7) root lattice is a sublattice of the f_4 root lattice. We can use su(2)-algebra's with elements $h_{\beta_i}, e_{\pm\beta_i}$, with $\beta_1 = (1,-1,0,0)/\sqrt{18}$, $\beta_2 = (1,1,0,0)/\sqrt{18}$, and $\beta_3 = (0,0,1,0)/\sqrt{18}$. The two diagonal subgroups and holonomies are constructed in the standard way.

$$\begin{aligned}
L_3^D &= l_3^1 + l_3^2 + l_3^3 = \sqrt{18} h_{(1,0,1,0)} \\
L_+^D &= l_+^1 + l_+^2 + l_+^3 = 3e_{(1,-1,0,0)/\sqrt{18}} + 3e_{(1,1,0,0)/\sqrt{18}} + \sqrt{18} e_{(0,0,1,0)/\sqrt{18}}
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
L_3^{D'} &= l_3^1 + l_3^2 + l_3^3 = \sqrt{18} h_{(0,1,1,0)} \\
L_+^{D'} &= l_+^1 + l_+^2 + l_+^3 = 3e_{-(1,-1,0,0)/\sqrt{18}} + 3e_{(1,1,0,0)/\sqrt{18}} + \sqrt{18} e_{(0,0,1,0)/\sqrt{18}}\,.
\end{aligned} \tag{3.29}$$

Again these are found to be so(3)-algebra's. The holonomies are as in (3.23). Calculating the (connected component of the) unbroken subgroup, one finds that it is SO(3). This can be understood from the branching of F_4 into $G_2 \times \mathrm{SO}(3)$.

$$\begin{aligned}
F_4 &\to G_2 \times \mathrm{SO}(3) \\
\mathbf{26} &\to (\mathbf{7},\mathbf{3}) \oplus (\mathbf{1},\mathbf{5}) \\
\mathbf{52} &\to (\mathbf{14},\mathbf{1}) \oplus (\mathbf{7},\mathbf{5}) \oplus (\mathbf{1},\mathbf{3})\,.
\end{aligned} \tag{3.30}$$

JHEP05(1999)001

Note that here it is important that our construction fits into G_2. Starting from SO(7) might lead to the wrong expectation that the unbroken subgroup would be SO(2), since F_4 branches into SO(7) $\times$ SO(2).

Case of E_6. In E_6 we use that the so(8) root lattice is a sublattice of the e_6 root lattice. We use su(2)-algebra's with elements $h_{\beta_i}, e_{\pm\beta_i}$, with $\beta_1 = (0,1,-1,0,0)/\sqrt{24}$, $\beta_2 = (0,1,1,0,0,0)/\sqrt{24}$, and $\beta_3 = (0,0,0,1,-1,0)/\sqrt{24}$, and $\beta_4 = (0,0,0,1,1,0)/\sqrt{24}$. The two diagonal subgroups are:

$$\begin{aligned}
L_3^D &= l_3^1 + l_3^2 + l_3^3 = \sqrt{24}h_{(0,1,0,1,0,0)} \\
L_+^D &= l_+^1 + l_+^2 + l_+^3 = \sqrt{12}\Big(e_{(0,1,-1,0,0,0)/\sqrt{24}} + e_{(0,1,1,0,0,0)/\sqrt{24}} + \\
&\qquad + e_{(0,0,0,1,-1,0)/\sqrt{24}} + e_{(0,0,0,1,1,0)/\sqrt{24}}\Big)
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
L_3^{D'} &= l_3^1 + l_3^2 + l_3^3 = \sqrt{24}h_{(0,0,1,1,0,0)} \\
L_+^D &= l_+^1 + l_+^2 + l_+^3 = \sqrt{12}\Big(e_{-(0,1,-1,0,0,0)/\sqrt{24}} + e_{(0,1,1,0,0,0)/\sqrt{24}} + \\
&\qquad + e_{(0,0,0,1,-1,0)/\sqrt{24}} + e_{(0,0,0,1,1,0)/\sqrt{24}}\Big)\,.
\end{aligned} \tag{3.32}$$

These are so(3)-algebra's. The holonomies are as in (3.23). The (connected component of the) unbroken subgroup is found to be SU(3). This can be understood from the branching of E_6 into $G_2 \times$ SU(3).

$$\begin{aligned}
&E_6 \to G_2 \times \mathrm{SU}(3) \\
&\mathbf{27} \to (\mathbf{7}, \bar{\mathbf{3}}) \oplus (\mathbf{1}, \mathbf{6}) \\
&\mathbf{78} \to (\mathbf{14}, \mathbf{1}) \oplus (\mathbf{7}, \mathbf{8}) \oplus (\mathbf{1}, \mathbf{8})\,.
\end{aligned} \tag{3.33}$$

Case of E_7. The E_6-example can be trivially embedded in E_7, using that the e_6 root lattice is a sublattice of the e_7 root lattice. We add a zero to the vectors and adapt the normalisations.

$$\begin{aligned}
L_3^D &= l_3^1 + l_3^2 + l_3^3 = 6h_{(0,1,0,1,0,0,0)} \\
L_+^D &= l_+^1 + l_+^2 + l_+^3 = \sqrt{18}\big(e_{(0,1,-1,0,0,0,0)/6} + e_{(0,1,1,0,0,0,0)/6} + \\
&\qquad + e_{(0,0,0,1,-1,0,0)/6} + e_{(0,0,0,1,1,0,0)/6}\big)
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
L_3^{D'} &= l_3^1 + l_3^2 + l_3^3 = 6h_{(0,0,1,1,0,0)} \\
L_+^D &= l_+^1 + l_+^2 + l_+^3 = \sqrt{18}\big(e_{-(0,1,-1,0,0,0,0)/6} + e_{(0,1,1,0,0,0,0)/6} + \\
&\qquad + e_{(0,0,0,1,-1,0,0)/6} + e_{(0,0,0,1,1,0,0)/6}\big)\,.
\end{aligned} \tag{3.35}$$

These are so(3)-algebra's. The holonomies are as in (3.23). The (connected component of the) unbroken subgroup is a simple group of 21 generators. It takes a little more work to show that it is[4] Sp(3) (and not SO(7)). This can be understood from the branching of E_7 into[5] $G_2 \times \mathrm{Sp}(3)$

$$\begin{aligned} E_7 &\to G_2 \times \mathrm{Sp}(3) \\ \mathbf{56} &\to (\mathbf{7},\mathbf{6}) \oplus (\mathbf{1},\mathbf{14}) \\ \mathbf{133} &\to (\mathbf{14},\mathbf{1}) \oplus (\mathbf{7},\mathbf{14'}) \oplus (\mathbf{1},\mathbf{21})\,. \end{aligned} \tag{3.36}$$

Case of E_8. The e_8 root lattice contains as a sublattice the e_7 root lattice. Again we add a zero to the vectors and adapt the normalisations.

$$\begin{aligned} L_3^D = l_3^1 + l_3^2 + l_3^3 &= \sqrt{60} h_{(0,1,0,1,0,0,0,0)} \\ L_+^D = l_+^1 + l_+^2 + l_+^3 &= \sqrt{30}\Big(e_{(0,1,-1,0,0,0,0,0)/\sqrt{60}} + e_{(0,1,1,0,0,0,0,0)/\sqrt{60}} + \\ &\qquad + e_{(0,0,0,1,-1,0,0,0)/\sqrt{60}} + e_{(0,0,0,1,1,0,0,0)/\sqrt{60}}\Big) \end{aligned} \tag{3.37}$$

$$\begin{aligned} L_3^{D'} = l_3^1 + l_3^2 + l_3^3 &= \sqrt{60} h_{(0,0,1,1,0,0,0)} \\ L_+^D = l_+^1 + l_+^2 + l_+^3 &= \sqrt{30}\Big(e_{-(0,1,-1,0,0,0,0,0)/\sqrt{60}} + e_{(0,1,1,0,0,0,0,0)/\sqrt{60}} + \\ &\qquad + e_{(0,0,0,1,-1,0,0,0)/\sqrt{60}} + e_{(0,0,0,1,1,0,0,0)/\sqrt{60}}\Big)\,. \end{aligned} \tag{3.38}$$

These are so(3)-algebra's. The holonomies are as in (3.23). The (connected component of the) unbroken subgroup is F_4. This can be understood from the branching of E_8 into $G_2 \times F_4$.

$$\begin{aligned} E_8 &\to G_2 \times F_4 \\ \mathbf{248} &\to (\mathbf{14},\mathbf{1}) \oplus (\mathbf{7},\mathbf{26}) \oplus (\mathbf{1},\mathbf{52})\,. \end{aligned} \tag{3.39}$$

4. Non-trivial vacua based on twist in SU($N > 2$)

Although we will publish the computational details in a separate article [5], we mention that our construction, with $\tilde{G}' = \mathrm{SU}(3), \mathrm{SU}(4), \mathrm{SU}(5)$ and $\mathrm{SU}(2) \times \mathrm{SU}(3)$ can be used for the exceptional groups other than G_2. A construction based on SU(3) is possible in F_4 and $E_{6,7,8}$, the SU(4) construction is possible in $E_{7,8}$, and E_8 allows an SU(5) based construction. Finally, using the $G_2 \times F_4$ subgroup of E_8, the SU(2) and SU(3) constructions can be embedded simultaneously in this group. The SU(3)-construction will yield two inequivalent vacuum components, obtained by taking n in eq. (2.12) 1 or 2 respectively. The SU(4) construction yields three different vacuum components, but, one of these vacuum components is already contained in

[4]Some would denote what we call Sp(n) as Sp($2n$). For our conventions e.g. $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$, $\mathrm{Sp}(2)/\mathbb{Z}_2 \cong \mathrm{SO}(5)$

[5]In the decompositions given, both 14-dimensional representations of Sp(3) are present. By **14**, we denote the representation with Dynkin labels (001), while **14′** is the irrep with Dynkin labels (010) (the simple roots of Sp(3) are ordered such that the longest one appears on the right) [9].

the SU(2)-construction described in this paper. The SU(5)-construction gives four different vacuum components. SU(2)$\times$SU(3) gives five different vacuum components, of which one is contained in our SU(2) construction, and two are contained in our SU(3)-construction, so only two vacua are new. We will label these different vacua by an integer related to the centre of the appropriate SU(N) embeddings required for twisting. Hence the integer is taken to be N for the SU(N) based construction, and 6 for the SU(2) $\times$ SU(3) construction. Obviously we reserve the label 1 for the trivial component.

The essence of the SU(2) based construction is the existence of a suitable subgroup PG'_N within a G_2 subgroup of G. The non-trivial flat connections arise through the decomposition of G into $G_2 \times H$, with H the maximal subgroup commuting with G_2. It is the CSA of H that determines the deformations (see the discussion below eq. (2.2)) of the non-trivial G_2 connection as embedded in G, fixing the dimension of this connected vacuum component (rank(H)). Since H commutes with the G_2 subgroup containing the holonomies, it also plays the role of the maximal unbroken subgroup, apart from some global discrete symmetries. Compare the situation to Witten's D-brane construction [3].

Our construction presented here goes via the chains

$$\begin{array}{ccccccc}
E_8 \to E_7 \to E_6 & & & & & & \\
& \searrow & & & & & \\
& \mathrm{SO}(2N) \to \mathrm{SO}(8) & & & & & \\
& & \searrow & & & & \\
& F_4 \to & \mathrm{SO}(7) & \to G_2 & & & \\
& & \nearrow & & & & \\
& \mathrm{SO}(2N+1) & & & & &
\end{array} \tag{4.1}$$

It is this general feature that repeats itself for the SU($N > 2$) based constructions. For SU(3) the role of G_2 is played by F_4, with the embedding chain $E_8 \to E_7 \to E_6 \to F_4$. For SU(4) the role of G_2 is played by E_7, with chain $E_8 \to E_7$, and finally for SU(5) and SU(2)$\times$SU(3), E_8 stands alone. Our results are summarised in table 1, which presents the connected component of the maximal unbroken subgroup for each vacuum-type.

So far we have concentrated on the classical gauge fields. For a calculation of the Witten index $\mathrm{Tr}(-)^F$, these should be quantised, and the fermions should be included. The computation is essentially the same as in [1, 3]: Each vacuum component implies a unique bosonic vacuum, and fermions can be added in $r'+1$ ways, where r' is the dimension of the vacuum component, which is equal to the rank of the unbroken subgroup. Thus each vacuum component contributes $r'+1$ to $\mathrm{Tr}(-)^F$. In table 2, we list $r'+1$, where r' is the rank of the group listed in table 1. The columns labeled by $3, 4, 5, 6$ contain more entry's to indicate that there is more than one vacuum component for each type. Finally, table 2 also lists the dual Coxeter number for each group. It is easy to verify that, rather miraculously, (1.1) is satisfied.

JHEP05(1999)001

Group G	Vacuum-type 1	2	3	4	5	6
$\mathrm{SU}(N)$	$\mathrm{SU}(N)$					
$\mathrm{Sp}(N)$	$\mathrm{Sp}(N)$					
$\mathrm{SO}(2N+1)$	$\mathrm{SO}(2N+1)$	$\mathrm{SO}(2N-6)$				
$\mathrm{SO}(2N)$	$\mathrm{SO}(2N)$	$\mathrm{SO}(2N-7)$				
G_2	G_2	discrete				
F_4	F_4	$\mathrm{SO}(3)$	discrete			
E_6	E_6	$\mathrm{SU}(3)$	discrete			
E_7	E_7	$\mathrm{Sp}(3)$	$\mathrm{SU}(2)$	discrete		
E_8	E_8	F_4	G_2	$\mathrm{SU}(2)$	discrete	discrete

Table 1: The connected part of the maximal unbroken subgroups.

Group G	h	Vacuum-type 1	2	3	4	5	6
$\mathrm{SU}(N)$	N	N					
$\mathrm{Sp}(N)$	$N+1$	$N+1$					
$\mathrm{SO}(2N+1)$	$2N-1$	$N+1$	$N-2$				
$\mathrm{SO}(2N)$	$2N-2$	$N+1$	$N-3$				
G_2	4	3	1				
F_4	9	5	2	(1+1)			
E_6	12	7	3	(1+1)			
E_7	18	8	4	(2+2)	(1+1)		
E_8	30	9	5	(3+3)	(2+2)	(1+1+1+1)	(1+1)

Table 2: Contributions to $\mathrm{Tr}(-1)^F$.

5. Conclusions

We have rederived the results of [3, 4], and embedded these results into the exceptional groups F_4 and $E_{6,7,8}$. The basic structure is contained in the exceptional group G_2, and the geometry (in a convenient gauge) is basically that of so-called multi-twisted boundary conditions. For G_2 and $\mathrm{SO}(N)$ our methods are in some sense complementary to those of [3, 4], some aspects are more conveniently described in our construction, other aspects are more transparant in the original approach. For the remaining exceptional groups an approach based on M-theory is not available (yet).

The contributions to $\mathrm{Tr}(-)^F$ from the embedding of the G_2 non-trivial flat connection in the exceptional groups F_4 and $E_{6,7,8}$ is still insufficient to satisfy (1.1). The generalisation of our method to $\mathrm{SU}(N)^M$ subgroups with $N > 2$ will allow us to derive additional non-trivial flat connections for these exceptional groups. In a subsequent publication [5] we will construct these extra vacua, and demonstrate that they can solve the Witten index problem for the exceptional groups.

JHEP05(1999)001

Acknowledgments

We would like to thank Pierre van Baal, Jan de Boer, Robbert Dijkgraaf, Arkady Vainshtein and Andrei Smilga for helpful discussions.

A. Lie algebras: conventions

Let $\mathcal{L}$ be a Lie algebra. The Killing form is given by:

$$B(a,b) = \mathrm{tr}\{\mathbf{ad}(a)\mathbf{ad}(b)\}\,. \tag{A.1}$$

A Cartan subalgebra $\mathcal{H}$ is a maximal abelian subalgebra in $\mathcal{L}$. Because all elements in $\mathcal{H}$ commute, they can be simultaneously diagonalised. In particular, in the adjoint representation, the eigenvectors of $\mathcal{H}$ are elements of the Lie algebra. One can write:

$$[h, e_\alpha] = \alpha(h) e_\alpha \quad h \in \mathcal{H}\,. \tag{A.2}$$

The $\alpha(h)$ are linear functionals on the space $\mathcal{H}$, called roots or root vectors. It is possible to associate elements of $\mathcal{H}$ to the functionals $\alpha(h)$ by defining

$$B(h_\alpha, h) = \alpha(h)\,. \tag{A.3}$$

Because of linearity one has

$$h_{p\alpha+q\beta} = p h_\alpha + q h_\beta\,. \tag{A.4}$$

The space $\mathcal{H}^*$ of linear functionals is a vector space. The roots form a (finite) subset of this space. The set of root vectors will be denoted by Δ. Because the Killing form is symmetric and bilinear one can introduce the notation

$$B(h_\alpha, h_\beta) = \alpha(h_\beta) = \beta(h_\alpha) \equiv \langle \alpha, \beta \rangle\,. \tag{A.5}$$

For compact Lie algebra's, the Killing form defines an inner product on the root space. The normalisation is determined by a self-consistency condition:

$$\langle \alpha, \alpha \rangle = B(h_\alpha, h_\alpha) = \sum_{\gamma \in \Delta} \langle \alpha, \gamma \rangle \langle \gamma, \alpha \rangle\,. \tag{A.6}$$

We now have

$$[h_\alpha, e_\beta] = \langle \alpha, \beta \rangle e_\beta\,. \tag{A.7}$$

The e_β are normalised such that

$$[e_\alpha, e_{-\alpha}] = 2h_\alpha\,. \tag{A.8}$$

If $\alpha + \beta \neq 0$:

$$[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta} \tag{A.9}$$

(We will never need an explicit form of $N_{\alpha,\beta}$). Hermitean conjugation acts as follows in our conventions

$$\begin{aligned} h_\alpha^\dagger &= h_\alpha\,, \\ e_\alpha^\dagger &= e_{-\alpha}\,. \end{aligned} \tag{A.10}$$

One picks a (non-orthogonal) basis of roots α_i such that, if α_i, α_j are in this basis, $\alpha_i - \alpha_j$ is not a root. The roots of such a basis are called "simple". Any root is expressible as $\sum_k c_k \alpha_k$, where the c_k are integers. We always denote simple roots by α_i, where i is an index or a number. The simple roots of the compact simple Lie algebra's are listed in appendix B.

Because the elements of the CSA h_α always commute, they can be simultaneously diagonalised in any matrix representation. In a specific matrix representation $(h_\alpha)_{ij}$, weights λ_i are defined by $(h_\alpha)_{ii} = \langle \alpha, \lambda_i \rangle$ for each α. Consequently the number of weights of a representation is equal to its dimension. A weight λ of a group is always of the form

$$\lambda = \sum_i n_i \Lambda_i + m_i \alpha_i\,, \qquad n_i, m_i \;\in \mathbb{Z}\,, \tag{A.11}$$

where Λ_i are the fundamental weights, and α_i the simple roots. The fundamental weights are defined from the simple roots by

$$2\frac{\langle \Lambda_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}\,. \tag{A.12}$$

The fundamental weights are always of the form $\sum_k q_k \alpha_k$ where the q_k are rational numbers.

B. Lie algebras: roots and weights

For easy refence we give some quantities for the groups used in this article. Our conventions are basically those of [10], where they can be found in appendix F. For G_2, we find it easier to work with abstract root vectors, for the orthogonal and other exceptional groups it is more convenient to work with explicit forms for the root vectors. We omit $\mathrm{Sp}(N)$, since we will never need it for explicit calculations. E_8, F_4 and G_2 do not possess non-integer fundamental weights, and hence none are listed. All fundamental weights of $\mathrm{SU}(N)$ are non-integer, but we will not need them, and hence they are not listed either.

We use the notation e_i for the unit vector in the i-direction. Inproducts $\langle \alpha, \beta \rangle$ that are not listed can either be obtained by $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle$, or are 0. In the non-simply laced algebra's, the solid dots in the Dynkin diagrams denote the shorter roots.

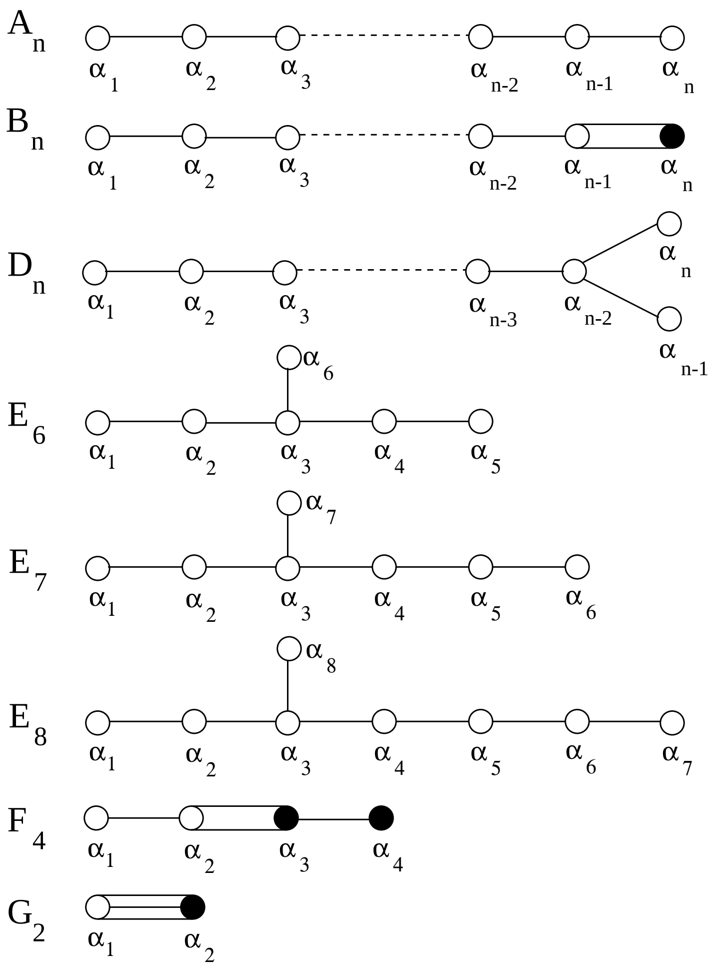

Figure 3: Dynkin diagrams

Roots for SU(n) **(**A_n**)**

- Simple roots:

$$\langle\alpha_i,\alpha_i\rangle = \tfrac{1}{n+1} \qquad (1 \le i \le n)$$
$$\langle\alpha_i,\alpha_{i+1}\rangle = -\tfrac{1}{2(n+1)} \qquad (1 \le (i+1) \le n)$$

- Positive roots:

$$\sum_{p=j}^{k}\alpha_p \qquad (1 \le j \le k \le n)$$

Roots and weights for $\mathrm{SO}(2n+1)$ **(B_n)**

- Simple roots:

$$\begin{aligned}
\langle\alpha_i,\alpha_i\rangle &= \frac{1}{2n-1} && (1\le i\le n-1)\\
\langle\alpha_n,\alpha_n\rangle &= \frac{1}{2(2n-1)} && \\
\langle\alpha_i,\alpha_{i+1}\rangle &= -\frac{1}{2(2n-1)} && (1\le (i+1)\le n)\,,
\end{aligned}$$

explicit representation:

$$\begin{aligned}
\alpha_i &= \frac{(e_i-e_{i+1})}{\sqrt{2(2n-1)}} && (1\le i< n)\\
\alpha_n &= \frac{e_n}{\sqrt{2(2n-1)}}\,.
\end{aligned}$$

- Positive roots:

$$\frac{e_k}{\sqrt{2(2n-1)}}\,,\qquad \frac{(e_k\pm e_l)}{\sqrt{2(2n-1)}}\,,\qquad (k<l)\,.$$

- Non-integer fundamental weights:

$$\Lambda_n=\frac{1}{\sqrt{2(2n-1)}}\sum_{i=1}^{n}e_i/2\,.$$

Roots and weights for $\mathrm{SO}(2n)$ **(D_n)**

- Simple roots:

$$\begin{aligned}
\langle\alpha_i,\alpha_i\rangle &= \frac{1}{2(n-1)} && (1\le i\le n)\\
\langle\alpha_i,\alpha_{i+1}\rangle &= -\frac{1}{4(n-1)} && (1\le (i+1)\le n-3)\\
\langle\alpha_{n-2},\alpha_{n-1}\rangle &= \langle\alpha_{n-2},\alpha_n\rangle=-\frac{1}{4(n-1)}\,,
\end{aligned}$$

explicit representation:

$$\begin{aligned}
\alpha_i &= \frac{(e_i-e_{i+1})}{\sqrt{4(n-1)}} && (1\le i< n)\\
\alpha_n &= \frac{(e_{n-1}+e_n)}{\sqrt{4(n-1)}}\,.
\end{aligned}$$

- Positive roots:

$$\frac{(e_k \pm e_l)}{\sqrt{4(n-1)}} \qquad (k < l)\,.$$

- Non-integer fundamental weights:

$$\begin{aligned}
\Lambda_{n-1} &= \frac{1}{\sqrt{2(n-1)}}\left(\sum_{i=1}^{n-2}\frac{e_i}{2} - \frac{e_{n-1}}{2} - \frac{e_n}{2}\right)\\
\Lambda_n &= \frac{1}{\sqrt{2(n-1)}}\left(\sum_{i=1}^{n-2}\frac{e_i}{2} - \frac{e_{n-1}}{2} + \frac{e_n}{2}\right)\,.
\end{aligned}$$

Roots and weights for E_6

- Simple roots:

$$\begin{aligned}
\langle\alpha_i,\alpha_i\rangle &= \frac{1}{12} \qquad\qquad (1 \le i \le 6)\\
\langle\alpha_1,\alpha_2\rangle &= \langle\alpha_2,\alpha_3\rangle = \langle\alpha_3,\alpha_4\rangle = -\frac{1}{24}\\
\langle\alpha_4,\alpha_5\rangle &= \langle\alpha_3,\alpha_6\rangle = -\frac{1}{24}\,,
\end{aligned}$$

explicit representation:

$$\begin{aligned}
\alpha_1 &= \frac{e_1\sqrt{3} - e_2 - e_3 - e_4 - e_5 - e_6}{2\sqrt{24}}\\
\alpha_2 &= \frac{e_5 + e_6}{\sqrt{24}}\\
\alpha_3 &= \frac{e_4 - e_5}{\sqrt{24}}\\
\alpha_4 &= \frac{e_3 - e_4}{\sqrt{24}}\\
\alpha_5 &= \frac{e_2 - e_3}{\sqrt{24}}\\
\alpha_6 &= \frac{e_5 - e_6}{\sqrt{24}}\,.
\end{aligned}$$

- Positive roots:

$$\frac{e_k \pm e_l}{\sqrt{24}} \quad (1 < k < l)\,, \qquad \frac{e_1\sqrt{3} \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6}{2\sqrt{24}}\,.$$

In the last expression the number of minus-signs should be odd.

- Non-integer fundamental weights:

$$\begin{aligned}
\Lambda_1 &= \frac{\frac{2}{3}\sqrt{3}e_1}{\sqrt{24}} \\
\Lambda_2 &= \frac{\frac{5}{6}\sqrt{3}e_1 + \frac{1}{2}(e_2 + e_3 + e_4 + e_5 + e_6))}{\sqrt{24}} \\
\Lambda_4 &= \frac{\frac{2}{3}\sqrt{3}e_1 + e_2 + e_3}{\sqrt{24}} \\
\Lambda_5 &= \frac{\frac{1}{3}\sqrt{3}e_1 + e_2}{\sqrt{24}} \,.
\end{aligned}$$

Roots and weights for E_7

- Simple roots:

$$\begin{aligned}
\langle \alpha_i, \alpha_i \rangle &= \frac{1}{18} \qquad\qquad (1 \le i \le 7) \\
\langle \alpha_1, \alpha_2 \rangle &= \langle \alpha_2, \alpha_3 \rangle = \langle \alpha_3, \alpha_4 \rangle = -\frac{1}{36} \\
\langle \alpha_4, \alpha_5 \rangle &= \langle \alpha_5, \alpha_6 \rangle = \langle \alpha_3, \alpha_7 \rangle = -\frac{1}{36}
\end{aligned}$$

explicit representation:

$$\begin{aligned}
\alpha_1 &= \frac{e_1\sqrt{2} - e_2 - e_3 - e_4 - e_5 - e_6 - e_7}{12} \\
\alpha_2 &= \frac{e_6 + e_7}{6} \\
\alpha_3 &= \frac{e_5 - e_6}{6} \\
\alpha_4 &= \frac{e_4 - e_5}{6} \\
\alpha_5 &= \frac{e_3 - e_4}{6} \\
\alpha_6 &= \frac{e_2 - e_3}{6} \\
\alpha_7 &= \frac{e_6 - e_7}{6}
\end{aligned}$$

- Positive roots:

$$\frac{e_1}{\sqrt{18}} \quad \frac{e_k \pm e_l}{6} \quad (1 < k < l) \qquad \frac{e_1\sqrt{2} \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7}{12}$$

In the last expression the number of minus-signs should be even.

- Non-integer fundamental weights:

$$\begin{aligned}
\Lambda_4 &= \frac{\frac{3}{2}\sqrt{2}e_1 + e_2 + e_3 + e_4}{6} \\
\Lambda_6 &= \frac{\frac{1}{2}\sqrt{2}e_1 + e_2}{6} \\
\Lambda_7 &= \frac{\sqrt{2}e_1 + \frac{1}{2}(e_2 + e_3 + e_4 + e_5 + e_6 - e_7)}{6}
\end{aligned}$$

Roots for E_8

- Simple roots:

$$\begin{aligned}
\langle\alpha_i,\alpha_i\rangle &= \frac{1}{30} \qquad\qquad (1\le i\le 8)\\
\langle\alpha_1,\alpha_2\rangle &= \langle\alpha_2,\alpha_3\rangle = \langle\alpha_3,\alpha_4\rangle = \langle\alpha_4,\alpha_5\rangle = -\frac{1}{60}\\
\langle\alpha_5,\alpha_6\rangle &= \langle\alpha_6,\alpha_7\rangle = \langle\alpha_3,\alpha_7\rangle = -\frac{1}{60}\,,
\end{aligned}$$

explicit representation:

$$\begin{aligned}
\alpha_1 &= \frac{e_1-e_2-e_3-e_4-e_5-e_6-e_7-e_8}{2\sqrt{60}}\\
\alpha_2 &= \frac{e_7+e_8}{\sqrt{60}}\\
\alpha_3 &= \frac{e_6-e_7}{\sqrt{60}}\\
\alpha_4 &= \frac{e_5-e_6}{\sqrt{60}}\\
\alpha_5 &= \frac{e_4-e_5}{\sqrt{60}}\\
\alpha_6 &= \frac{e_3-e_4}{\sqrt{60}}\\
\alpha_7 &= \frac{e_2-e_3}{\sqrt{60}}\\
\alpha_8 &= \frac{e_7-e_8}{\sqrt{60}}\,.
\end{aligned}$$

- Positive roots:

$$\frac{e_k\pm e_l}{\sqrt{60}} \quad (k<l) \qquad\qquad \frac{e_1\pm e_2\pm e_3\pm e_4\pm e_5\pm e_6\pm e_7\pm e_8}{2\sqrt{60}}\,.$$

In the last expression the number of minus-signs should be odd.

Roots for F_4

- Simple roots:

$$\begin{aligned}
\langle\alpha_1,\alpha_1\rangle &= \langle\alpha_2,\alpha_2\rangle = \frac{1}{9}\\
\langle\alpha_3,\alpha_3\rangle &= \langle\alpha_4,\alpha_4\rangle = \frac{1}{18}\\
\langle\alpha_1,\alpha_2\rangle &= \langle\alpha_2,\alpha_3\rangle = -\frac{1}{18}\\
\langle\alpha_3,\alpha_4\rangle &= -\frac{1}{36}\,,
\end{aligned}$$

explicit representation:

$$\begin{aligned} \alpha_1 &= \frac{e_2 - e_3}{\sqrt{18}} \\ \alpha_2 &= \frac{e_3 - e_4}{\sqrt{18}} \\ \alpha_3 &= \frac{e_4}{\sqrt{18}} \\ \alpha_4 &= \frac{e_1 - e_2 - e_3 - e_4}{2\sqrt{18}} \end{aligned}$$

- Positive roots:

$$\frac{e_k}{\sqrt{18}} \qquad \frac{e_k \pm e_l}{\sqrt{18}} \qquad (k < l) \qquad \frac{e_1 \pm e_2 \pm e_3 \pm e_4}{2\sqrt{18}}$$

Roots for G_2

- Simple roots:

$$\langle \alpha_1, \alpha_1 \rangle = \frac{1}{4}, \qquad \langle \alpha_1, \alpha_2 \rangle = -\frac{1}{8}, \qquad \langle \alpha_2, \alpha_2 \rangle = \frac{1}{12}.$$

- Positive roots:

$$\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2 .$$

References

[1] E. Witten, *Nucl. Phys.* **B 202** (1982) 253.

[2] S.F. Cordes and M. Dine, *Nucl. Phys.* **B 273** (1986) 581;
A. Morozov, M. Ol'shanetsky and M. Shifman, *Nucl. Phys.* **B 304** (1988) 291.

[3] E. Witten, *J. High Energy Phys.* **02** (1998) 006 [hep-th/9712028].

[4] A. Keurentjes, A. Rosly and A.V. Smilga, *Phys. Rev.* **D 58** (1998) 081701 [hep-th/9805183].

[5] A. Keurentjes, *Non–trivial flat connections on the 3–torus II: The exceptional groups F_4 and $E_{6,7,8}$*, to be published.

[6] G.'t Hooft, *Nucl. Phys.* **B 153** (1979) 141.

[7] E. Cohen and C. Gomez, *Nucl. Phys.* **B 223** (1983) 183.

[8] E.B. Dynkin, *Mat. Sbornik* **30** (**72**) (1952) 349 (English translation: *Amer. Math. Soc. Trans. Series 2*, **6**, (1957) 111).

[9] W.G. McKay and J. Patera, *Tables of dimensions, indices, and branching rules for representations of simple Lie algebras*, Marcel Dekker, New York and Basel 1981.

[10] J.F. Cornwell, *Group theory in physics,* **Volume 2**, Academic Press, London 1984.

JHEP05(1999)001